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# On a class of rigid Coxeter groups(Cohomology Theory of Finite Groups and Related Topics)

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## On a class of rigid Coxeter groups

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The purpose of this note is to introduce some results of recent papers [4] and [5] about rigid Coxeter groups.

A *Coxeter group* is a group  $W$  having a presentation

$$\langle S \mid (st)^{m(s,t)} = 1 \text{ for } s, t \in S \rangle,$$

where  $S$  is a finite set and  $m : S \times S \rightarrow \mathbb{N} \cup \{\infty\}$  is a function satisfying the following conditions:

- (i)  $m(s, t) = m(t, s)$  for any  $s, t \in S$ ,
- (ii)  $m(s, s) = 1$  for any  $s \in S$ , and
- (iii)  $m(s, t) \geq 2$  for any  $s, t \in S$  such that  $s \neq t$ .

The pair  $(W, S)$  is called a *Coxeter system*. For a Coxeter group  $W$ , a generating set  $S'$  of  $W$  is called a *Coxeter generating set for  $W$*  if  $(W, S')$  is a Coxeter system. Let  $(W, S)$  be a Coxeter system. For a subset  $T \subset S$ ,  $W_T$  is defined as the subgroup of  $W$  generated by  $T$ , and called a *parabolic subgroup*. A subset  $T \subset S$  is called a *spherical subset of  $S$* , if the parabolic subgroup  $W_T$  is finite.

Let  $(W, S)$  and  $(W', S')$  be Coxeter systems. Two Coxeter systems  $(W, S)$  and  $(W', S')$  are said to be *isomorphic*, if there exists a bijection  $\psi : S \rightarrow S'$  such that

$$m(s, t) = m'(\psi(s), \psi(t))$$

for every  $s, t \in S$ , where  $m(s, t)$  and  $m'(s', t')$  are the orders of  $st$  in  $W$  and  $s't'$  in  $W'$ , respectively.

A *diagram* is an undirected graph  $\Gamma$  without loops or multiple edges with a map  $\text{Edges}(\Gamma) \rightarrow \{2, 3, 4, \dots\}$  which assigns an integer greater than 1 to each of its edges. Since such diagrams are used to define Coxeter systems, they are called *Coxeter diagrams*.

In general, a Coxeter group does not always determine its Coxeter system up to isomorphism. Indeed some counter-examples are known.

**Example** ([1, p.38 Exercise 8], [2]). It is known that for an odd number  $k \geq 3$ , the Coxeter groups defined by the diagrams in Figure 1 are isomorphic and  $D_{2k}$ .



FIGURE 1. Two distinct Coxeter diagrams for  $D_{2k}$

**Example** ([2]). It is known that the Coxeter groups defined by the diagrams in Figure 2 are isomorphic by the *diagram twisting* ([2, Definition 4.4]).

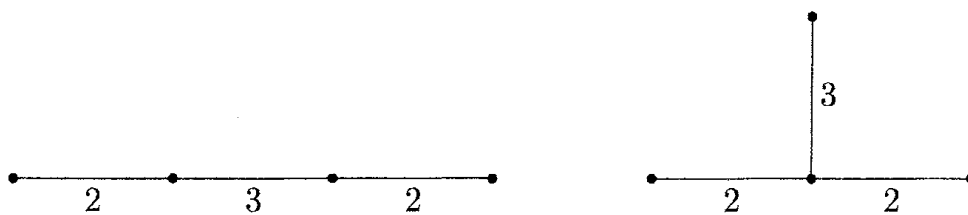


FIGURE 2. Coxeter diagrams for isomorphic Coxeter groups

Here there exists the following natural problem.

**Problem** ([2], [3]). When does a Coxeter group determine its Coxeter system up to isomorphism?

A Coxeter group  $W$  is said to be *rigid*, if the Coxeter group  $W$  determines its Coxeter system up to isomorphism (i.e., for each Coxeter generating sets  $S$  and  $S'$  for  $W$  the Coxeter systems  $(W, S)$  and  $(W, S')$  are isomorphic).

A Coxeter system  $(W, S)$  is said to be *even*, if  $m(s, t)$  is even for all  $s \neq t$  in  $S$ . Also a Coxeter system  $(W, S)$  is said to be *strong even*, if  $m(s, t) \in \{2\} \cup 4\mathbb{N}$  for all  $s \neq t$  in  $S$ .

The following theorem was proved by Radcliffe in [6].

**Theorem 1** ([6]). *If  $(W, S)$  is a strong even Coxeter system, then the Coxeter group  $W$  is rigid.*

In [4], we first proved the following theorem which give a new class of rigid Coxeter groups.

**Theorem 2.** *Let  $(W, S)$  be a Coxeter system. Suppose that*

- (0) *for each  $s, t \in S$  such that  $m(s, t)$  is even,  $m(s, t) = 2$ ,*
- (1) *for each  $s \neq t \in S$  such that  $m(s, t)$  is odd,  $\{s, t\}$  is a maximal spherical subset of  $S$ ,*
- (2) *there does not exist a three-points subset  $\{s, t, u\} \subset S$  such that  $m(s, t)$  and  $m(t, u)$  are odd, and*
- (3) *for each  $s \neq t \in S$  such that  $m(s, t)$  is odd, the number of maximal spherical subsets of  $S$  intersecting with  $\{s, t\}$  is at most two.*

*Then the Coxeter group  $W$  is rigid.*

**Example.** The Coxeter groups defined by the diagrams in Figure 3 are rigid by Theorem 2.

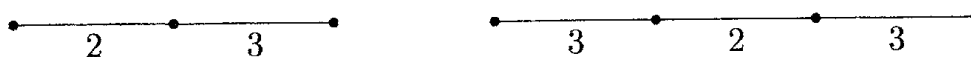


FIGURE 3. Coxeter diagrams for rigid Coxeter groups

In [5], we also proved the following theorem which is an extension of Theorem 1 and Theorem 2.

**Theorem 3.** *Let  $(W, S)$  be a Coxeter system. Suppose that*

- (0) *for each  $s, t \in S$  such that  $m(s, t)$  is even,  $m(s, t) \in \{2\} \cup 4\mathbb{N}$ ,*
- (1) *for each  $s \neq t \in S$  such that  $m(s, t)$  is odd,  $\{s, t\}$  is a maximal spherical subset of  $S$ ,*
- (2) *there does not exist a three-points subset  $\{s, t, u\} \subset S$  such that  $m(s, t)$  and  $m(t, u)$  are odd, and*
- (3) *for each  $s \neq t \in S$  such that  $m(s, t)$  is odd, the number of maximal spherical subsets of  $S$  intersecting with  $\{s, t\}$  is at most two.*

*Then the Coxeter group  $W$  is rigid.*

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